Newton's method for finding extrema

1. Use three steps of Newton's method to approximate a minima of the function $f(x) \stackrel{\text{def}}{=} \frac{\sin(x)}{x} + e^{-x}$ starting with $x_0 = 4.0$.

Answer: To ten significant digits, we have 4.0, 4.506177503, 4.542625973, 4.542956159

2. Given that the absolute error at one step of Newton's method is $|x_k - m|$ where m is a minimum of a function f, demonstrate that the absolute error $|x_{k+1} - m|$ is proportional to a value multiplied by $|x_k - m|^2$ under the assumption that $1 < |f^{(2)}(x)| < 2$ in the vicinity of the minimum, and $|f^{(3)}(x)| < 5$ in the vicinity of the minimum.

Answer: See the course notes.

3. Use three steps of Newton's method to approximate a minimum of the function $f(x) \stackrel{\text{def}}{=} x^4 - 6x^2 + 4x + 4$ starting with $x_0 = -0.6776507$

Answer: To ten significant digits, we have –0.6776507, 1.000000005, 33145927.27, 22097284.84.

4. What is the cause for the sequence of approximations in Question 3?

Answer: The function has a second derivative equal to zero when $x = 1$, so when x_1 is approximately equal to one, the reciprocal of the second derivative is very large.

5. If you continue to iterate Newton's method in Question 3, what minima does it converge to?

Answer: To ten significant digits, 1.532088886

6. If you iterated Newton's method in Question 3 but starting with $x_0 = -0.6$, what extremum does it converge to?

Answer: To ten significant digits, 0.3472963553, but this is a local maximum.

7. In general, should you apply Newton's method if you don't already have an idea as to what a minimum of a function is?

Answer: In general, no. Newton's method is a tool to refine an approximation of an extremum, not to check if a function has an extremum. If you start with an arbitrary initial point, it may or may not converge to an extremum if there is one, so non-convergence does not suggest there is no extremum.

8. The function x^4 has a double minimum at $x = 0$. Apply Newton's method starting with the initial value $x_0 = 1$. Is the convergence still $O(h^2)$?

Answer: No, with a double minimum, the rate of convergence becomes $O(h)$.